Non-linear Dynamics of the Damped and Driven Pendulum

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PHY 235W: Classical Mechanics Term Paper

The damped and driven pendulum was used to study non-linear dynamics and regions of chaotic motion. A model of a simple pendulum subject to a damping force and a driving torque about its pivot point was created using the generalized form of the Euler-Lagrange Equation. The driving torque of the pendulum was modelled using a cosine function with a variable amplitude and fixed angular frequency. Mathematical and numerical analyses of the dynamics of the pendulum were carried out in an attempt to understand the non-linearity and identify regions of chaotic behaviour. Regions of periodic oscillations, non-linear and chaotic oscillations were identified by varying the magnitude of the driving torque. The software Mathematica was used to solve this model numerically for different input parameter values and the results were visualized as time-domain and phase space evolutions as well as with Poincaré Sections.

I. INTRODUCTION

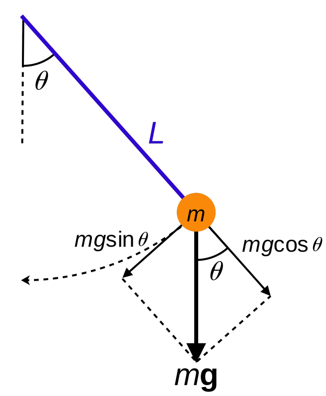
Determinism has been considered one of the strongest qualities of physics, especially within the realm of classical mechanics. The equations of motion derived for a particular system were expected to characterize completely the future evolution and behaviour of the system, given certain initial parameters. However, it was eventually discovered that certain systems, especially non-linear ones exhibited seemingly random behaviour. Systems whose time-evolution have a very sensitive dependence on initial conditions were said to exhibit deterministic chaos. Chaos is a phenomenon that occurs commonly in nature in the form of weather, and is essentially the result of a future state of a system being sensitive to its previous state. Thus, even a small change in initial conditions of the system can generate a time-evolution that is very different (and unique)

It is known that deterministic chaos is associated with non-linearity of a system since it is a necessary (but not sufficient) condition for chaotic behaviour to occur. The damped and driven pendulum is a relatively simple system that can be modelled and solved (numerically) to show certain non-linear and chaotic behaviour. This allows for understanding non-linear dynamics and showing how it is a necessary but not a sufficient condition for chaos to occur.

II. THEORETICAL BACKGROUND

The plane pendulum shown in Figure 1 below can be studied in a fairly straightforward manner, under the assumption that the oscillations are constrained to be small.

FIG. 1: The Plane Pendulum



A second order differential equation (1) of motion describes the angular motion of the pendulum as a function of time. Note that we conveniently define the constant $\omega_0^2 = \frac{g}{l}$ and using either Newtonian or Lagrangian mechanics we get the equation:

$$\frac{d^2\theta}{dt^2} + \omega_0^2 sin\theta = 0 \quad (1)$$

The solutions to this equation are non-linear since the values that θ can take are unconstrained. Using the kinetic and potential energy of the bob and knowledge about elliptic integrals (of the first kind), the solution for the period of the plane pendulum can be shown, to fourth order to be:

$$\tau = 2\pi\sqrt{\frac{l}{g}}[1 + \frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4] \quad (2)$$

It is easy to see that in the limit of small oscillations, when θ is restricted to small values we have $\theta = sin\theta$. Then equation (1) reduces to $\frac{d^2\theta}{dt^2} + \omega_0^2\theta = 0$. This is a relatively straightforward differential equation to solve, yielding the simple harmonic (sinusoidal) oscillator solutions. And we can also see in the limit that θ is small, the expression for the period of the pendulum from equation (2) above reduces to the following familiar expression.

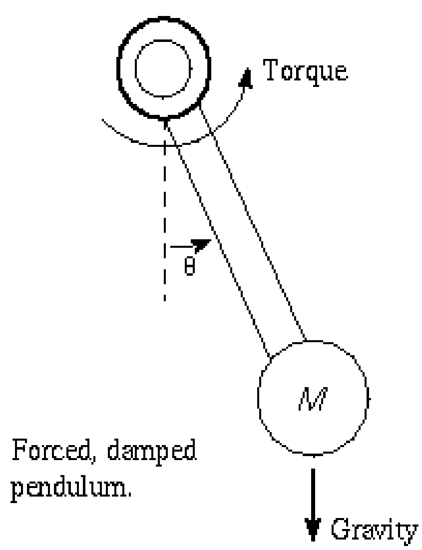
$$\tau = 2\pi\sqrt{\frac{l}{g}} \tag{3}$$

Geared with a firm understanding of oscillations of the plane pendulum and some basic concepts about non-linearity, we can model the damped and driven pendulum successfully and attempt to solve its equations of motion numerically.

III. DAMPED AND DRIVEN PENDULUM MODEL

For our analysis, a simple model of the damped and driven pendulum was developed. Note that the pendulum can be visualized as seen in Figure 2 below.

FIG. 2: The Damped and Driven Pendulum



In our model of the damped and driven pendulum, we shall be using the following definitions and basic conventions, some of which were applicable to the plane pendulum.

- The acceleration due to gravity is downwards, in the $-y$ direction and is defined by the constant g
- The pendulum of mass m is suspended by a mass-less rigid rod of length l
- $\theta(t)$ is a measure of the angular displacement of the pendulum from its equilibrium position
- The pendulum is suspended from a fixed pivot point, which is subject to a driving torque characterized by $T_d cos(\omega_d t)$. ω_d gives the angular frequency of the driving torque
- The Moment of Inertia of the Pendulum with respect to its pivot point is given by $I = ml^2$
- The pendulum is subject to a damping force characterized by the damping coefficient, b and the damping force is proportional to $\frac{d\theta}{dt}$
- There were no constraints placed on the values of θ. However, for the purpose of numerical analysis, values of b and ω_d were chosen carefully and are subsequently discussed.

The equation of motion for the damped and driven pendulum can be derived both using Newtonian Mechanics and using the Euler-Lagrange equation. We shall derive the equation using Lagrangian mechanics (and the Euler-Lagrange equation).

First we must define our coordinate system using polar coordinates in order to express our equation of motion in terms of the angular variable θ instead of Cartesian xy coordinates.

$$x = lsin\theta$$

$$y = -lcos\theta$$

Then we can calculate the required time derivatives to obtain expressions for $\dot{x}$ and $\dot{y}$

$$\dot{x} = lcos\theta\dot{\theta}$$

$$\dot{y} = lsin\theta\dot{\theta}$$

Using these expressions, the kinetic energy, T of the pendulum can be computed:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m[\dot{x}^2 + \dot{y}^2]$$

Substituting in the calculated values for $\dot{x}$ and $\dot{y}$ and using the identity $sin^2\theta + cos^2\theta = 1$, we get:

$$T = \frac{1}{2}ml^2\dot{\theta}^2 \tag{4}$$

Setting the equilibrium point of the mass at the bottom of the oscillations as $U = 0$, the potential energy, U of the pendulum bob is given by:

$$U = mgl(1 - cos\theta) \tag{5}$$

The Lagrangian thus obtained, $L = T - U$ is the same as that for the plane pendulum.

$$L = T - U = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - cos\theta) \tag{6}$$

However, one must add to the equation the additional constraints imposed by the damping and driving forces. The damping force is given by $F_b = -b\dot{\theta}$ and the driving torque is given by $F_d = T_d cos(\omega_d t)$.

We must use the generalized form of the Euler-Lagrange equation to include these additional forces. This form includes the generalized non-constraint forces of damping, F_b and driving, F_d that we have. Thus,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Q_i$$

where the generalized force,

$$Q_i = \sum_j F_j \frac{\partial s_j}{\partial q_j} = F_b + F_d$$

Including these forces for our model in the Euler Lagrange equation of motion given above we get:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} + F_b + F_d$$

$$\frac{d}{dt}\frac{\partial (T-U)}{\partial \dot{\theta}} = \frac{\partial (T-U)}{\partial \theta} - b\dot{\theta} + T_d cos(\omega_d t)$$

Calculating the required partial derivates, and simplifying we get a second order differential equation that describes the angular motion of the damped and driven pendulum we have modelled.

$$ml^2\ddot{\theta} = -mglsin\theta - b\dot{\theta} + T_d cos(\omega_d t) \tag{7}$$

This equation must be further simplified and manipulated so as to help simplify numerical calculations in a manner similar to Chapter 4 in [1]. This can be done by carefully substituting and introducing dimensionless parameters and quantities and simplifying the equation in the following manner.

First we divide both sides of the equation by ml^2 which is also an expression for the moment of inertia.

$$\ddot{\theta} + \frac{g}{l}sin\theta + \frac{b}{ml^2}\dot{\theta} = \frac{T_d}{ml^2}cos(\omega_d t)$$

Then we introduce the following parameters and make substitutions to use dimensionless time, t' and the variable $x = \theta$ for angular displacement:

$$c = \frac{b}{ml^2}$$

$$F = \frac{T_d}{ml^2}$$

$$\omega_0 = \sqrt{\frac{g}{l}} = \frac{1}{t_0}$$

$$\omega = \frac{\omega_d}{\omega_0} = \sqrt{\frac{l}{g}}\omega_d$$

$$t' = \frac{t}{t_0}$$

Using the substitution $x = \theta$ and the dimensionless time, $t' = \frac{t}{t_0}$ we get the following expressions for $\dot{x}$ and $\ddot{x}$ using chain rule:

$$\dot{x} = \frac{\dot{\theta}}{\omega_0}$$

$$\ddot{x} = \frac{\ddot{\theta}}{\omega_0^2}$$

The equation of motion for the damped and driven pendulum from equation (7) becomes:

$$\ddot{x}\omega_0^2 + \omega_0^2 sinx + \frac{b}{ml^2}\dot{x}\omega_0 = F\omega_0^2 cos(\omega_d t)$$

Dividing through by ω_0^2 and substituting in the necessary parameters, we get our simplified form of the equation of motion as a function of the new variable x and dimensionless time, t', to be used for numerical analysis.

$$\ddot{x} + sinx + c\dot{x} = Fcos(\omega t') \tag{8}$$

Note that this is a second order differential equation and is entirely characterized by the three parameters c, F and ω. This equation can also be written as a first order differential equation if we make the following substitutions:

$$y = \frac{dx}{dt'}$$

and

$$z = wt'$$

The new equation as a function of x, y and z, which is equivalent to (8) is:

$$\frac{dy}{dt'} = -cy - sinx + Fcos(z) \tag{9}$$

We thus see that equations (8) and (9) completely describe the damped and driven pendulum for our purpose of analysis of oscillations. In order to understand the behaviour of the pendulum and examine specific non-linear (and chaotic) regions, numerical techniques must be used to solve the differential equations described by (8) and (9). The subsequent section deals with the solutions to these equations for specific parameter values using numerical techniques.

IV. NUMERICAL ANALYSIS

The numerical analysis for the oscillations of the damped and driven pendulum was primarily carried out using code in Mathematica. The equation of motion derived in (8) above for the pendulum was solved using the *NDSolve* function in Mathematica and the results were evaluated by plotting the solutions for angular velocity both in the time-domain and in phase space.

The input parameters to the equation were varied carefully. The values for ω and c were held constant for all numerical solutions. This was done in an attempt to isolate the dependence of the non-linear behaviour of the driven pendulum on the magnitude of the driving torque, F. Thus the differential equation was solved for several values of the driving torque F, and the following characterizes the range of numerical solutions computed

- The values $\omega = 0.7$ and $c = 0.05$ were used and held constant for all solutions computed
- The value of the driving force, F was varied in intervals of approximately 0.1 in the range $0 < F < 1.2$. Larger values of the driving force, $F >> 1$ were also tested and solved for in an attempt to discover transient, non-linear and chaotic behaviour

A. Key Observations from Numerical Solutions

Based on careful observations of the numerically generated solutions for the various input parameters tested, we were able to observe definite non-linear periodic orbits and chaotic regions for the oscillations of the damped and driven pendulum. Note that all observations were made with respect to the angular velocity of the pendulum, both in time domain and in phase space.

1. Transient characteristics were observed for every solution generated and had to be ignored while plotting results to show periodic orbits. Figures 3 and 4 show the transient time-domain and phase space behaviour of the damped and driven pendulum for $F = 0.4$. From the time-domain plot is easy to observe that the system appears to oscillate in a seemingly chaotic manner. Similarly the parametric plot

FIG. 3: Transient Time Domain Evolution for F=0.4, $0 < t < 400$

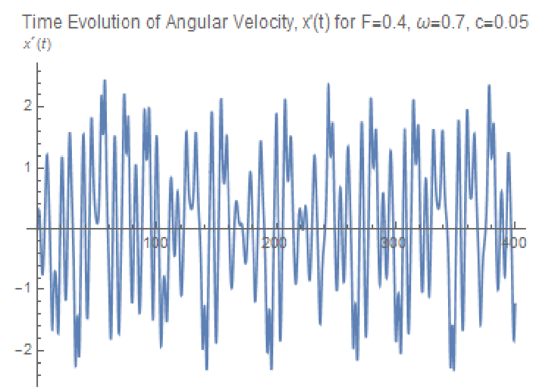
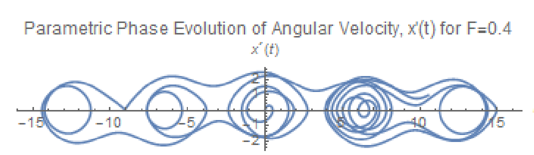


FIG. 4: Transient Parametric Phase Space Evolution for F=0.4, $0 < t < 50\pi$



2. It was interesting to note that the transient characteristics for the oscillations appeared to be seemingly aperiodic and chaotic, but for certain input parameter values - $F = 0.4, 0.5, 0.8, 0.9$ - these transient chaotic oscillations eventually settled into a steady periodic orbit. This non-linear periodic behaviour

can be seen in Figure 5 and 6 which show the time domain and parametrized phase space evolution for $F = 0.5$ after transients have died out.

FIG. 5: Periodic Time Domain Evolution for F=0.5, $2000 < t < 2200$

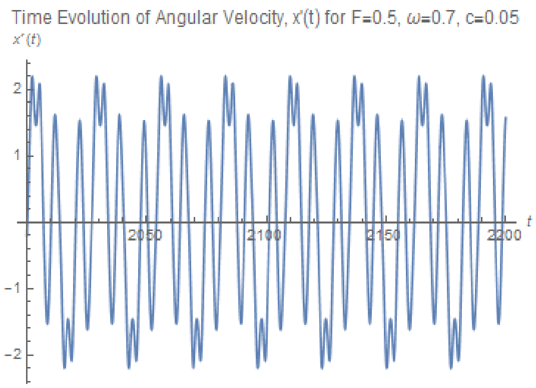
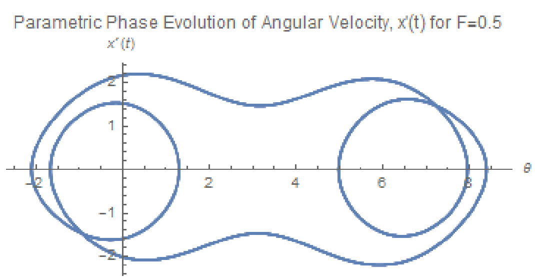


FIG. 6: Periodic Orbits: Parametric Phase Space Evolution for F=0.5, $2000\pi < t < 2100\pi$



We can see from the above figures for $F = 0.5$ that the angular velocity of the pendulum is complicated but evolves in a periodic manner with time. Furthermore, in phase space we can see that the angular velocity is constrained to a particular stable orbit (which corresponds to its periodic time-domain evolution).

3. It was observed from the numerical computations that the pendulum is in fact very sensitive to its initial conditions and regions of chaotic evolution of the angular velocity of the pendulum were discovered: for example input parameters of $F = 0.6 - 0.7$, $F = 0.92 - 1.4$ were shown to display chaotic behaviour. A visual example of this chaotic evolution of the system for $F = 0.65$ can be seen below in Figures 7 and 8 through the time domain and parametrized phase space evolution plots. Note again that these plots are generated after the transients died out, and it can be seen that the system continues to evolve in a chaotic (seemingly random) fashion. The phase space representation helps visualize the fact that the orbits are indeed non-periodic: The angular velocity, $x'(t)$ evolves without repeating its path in any periodic pattern over the entire time interval.

FIG. 7: Chaotic Time Domain Evolution for F=0.65, $2000 < t < 2200$

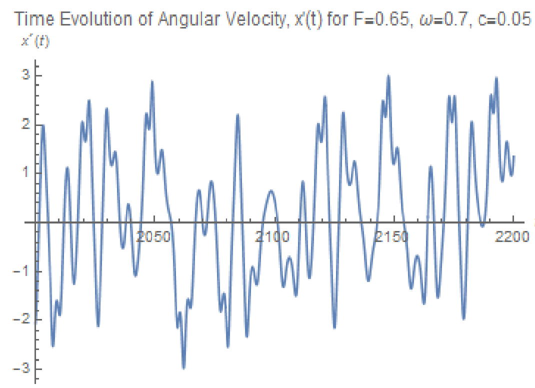
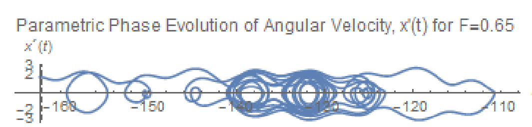


FIG. 8: Chaotic Phase Space Evolution for F=0.65, $2000\pi < t < 2100\pi$



4. Large values of F (such as $F = 200$) were shown to exhibit very minimal transient characteristics and the orbits were clearly seen to be periodic and non-chaotic. This is an expected result since in the limit that $F >> 1$, the driving torque, T_d dominates the equation of motion, reducing transient oscillations and forcing periodic behaviour. Figures 9 and 10 below help illustrate this with the time domain and phase space plots for $f = 200$.

FIG. 9: Stable Time Domain Evolution for F=200, $400 < t < 600$

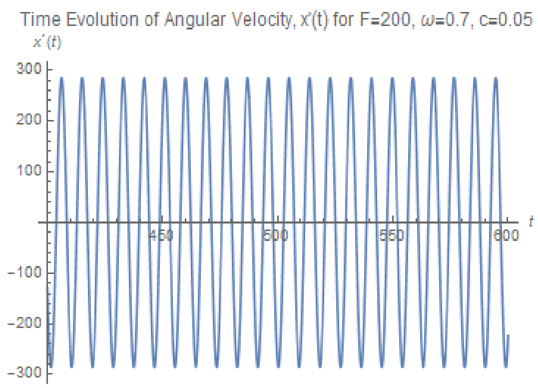
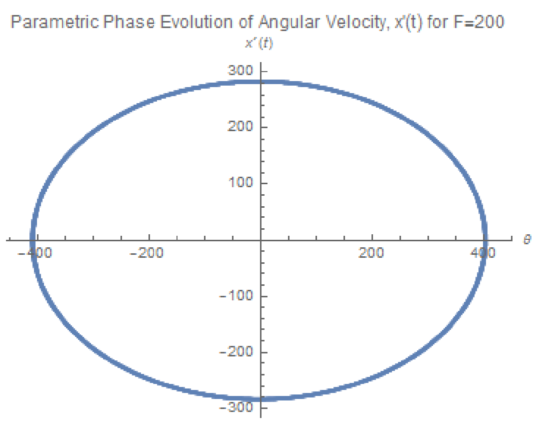


FIG. 10: Stable Phase Space Evolution for F=200, $400 < t < 500\pi$



5. Using Mathematica to observe the time domain and phase space evolution of the solutions to the damped and driven pendulum, it was possible to summarize the results in terms of the various periodic and chaotic regions identified. Note that these regions can be thought of as functions of the driving torque, F which was iterated through a range of values to generate several numerical solutions as described earlier.

   The following table shows the behaviour of the damped and driven pendulum for different solved values of the driving torque, F.

TABLE I: Periodic and Chaotic Regions of the Damped and Driven Pendulum

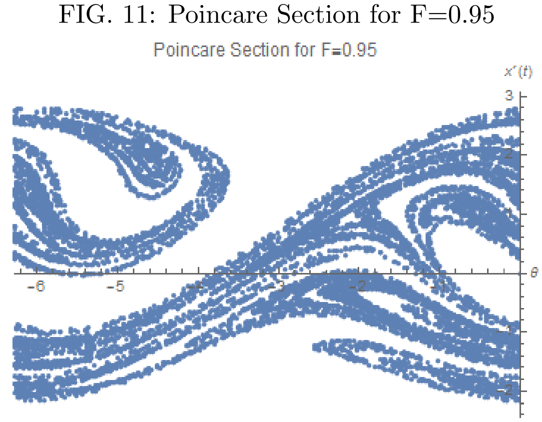
Driving Torque, F	**Evolution of $x'(t)$**	**Comments**
$0.1 < F < 0.4$	Periodic	Simple Harmonic Orbits
$F = 0.4$	Periodic	Simple Harmonic Orbit
$0.4 < F < 0.5$	Periodic	Stable Orbits
$F = 0.5$	Periodic	Stable Orbit
$0.55 < F < 0.75$	Chaotic	Aperiodic Orbits
$0.8 < F < 0.9$	Periodic	Stable Orbits
$0.95 < F < 1.4$	Chaotic	Aperiodic Orbits
$F > 1.4$	Periodic	Stable Orbits
$F >> 1$	Periodic	Stable Orbits, Low Transients

B. Poincaré Sections

Further analysis was conducted using Poincaré sections to study the dependence of the angular velocity periodically on time, by taking discrete snapshots of the angular velocity at every period, $\frac{2\pi}{\omega}$ of the driving force, F. These sections enable one to visualize the periodic behaviour of $x'(t)$ by taking stroboscopic (periodic) sections of the solution at periodic intervals of $z = \omega t'$.

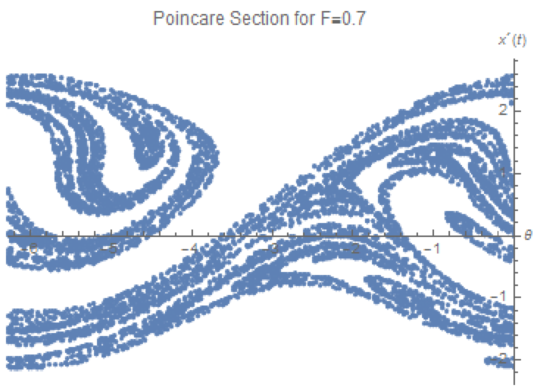
Thus, the following Poincaré sections essentially represent periodic solutions of $x'(t)$ as a function of θ plotted over a large time interval. Note that these plots were generated in Mathematica, assuming that θ could evolve in an unconstrained manner and thus there was no restriction for θ (and thus x and $\dot{x}$) to be bounded between specific values.

FIG. 11: Poincare Section for F=0.95



It can be seen from Figures 11 and 12 that the Poincareé sections for $F = 0.7$ and $F = 0.95$ are rich in structure for chaotic sections. The evolution of these chaotic sections is determined by *chaotic attractors* which are the reason behind the intricate patterns generated in these figures.

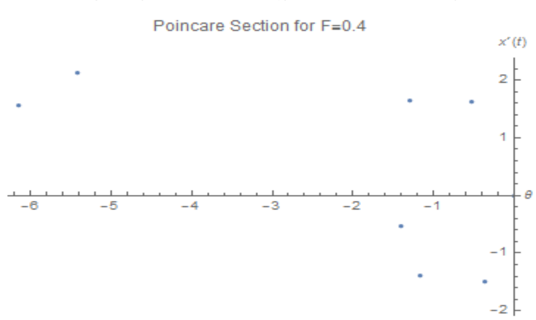
FIG. 12: Poincare Section for F=0.7



The chaotic evolution of $x'(t)$ is determined by the folding and stretching of bounded trajectories in phase space as the motion converges (often in a fractal pattern) due to attractor. These trajectories are continually diverging from each other but must return to the attractor [1].

Figure 13 below shows a Poincaré Section for $F = 0.4$ and we can observe the periodicity of the evolution of $x'(t)$ as it only takes up a few values periodically over a large interval of time. This can be easily contrasted with the chaotic sections where the evolution of $x'(t)$ was characterized by a varied and dense spread of values (in a bounded and patterned region).

FIG. 13: Poincare Section for F=0.4



V. CONCLUSION

A model of the damped and driven pendulum was created using the generalized form of the Euler-Lagrange Equation. This model was characterized by the differential equation, $\ddot{x} + sinx + c\dot{x} = Fcos(\omega t')$. Mathematica was used to solve this equation numerically for different input parameter values and the results were visualized as time-domain and phase space evolutions as well as Poincaré Sections.

Thus, the non-linear dynamics of the damped and driven pendulum were studied in some detail and observations regarding the behaviour of this pendulum were made. Regions of non-linearity, periodic oscillations and chaotic evolution were identified and analyzed in some detail.

VI. REFERENCES

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